Fundamentals of Kalman Filtering and Estimation in Aerospace Engineering

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Outline

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- Linear and Nonlinear Systems

Least Squares Estimation

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- Stochastic Processes
- The Kalman Filter Revealed

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Why Estimate?

- We estimate without even being conscious of it
- Anytime you walk down the hallway, you are estimating, your eyes and ears are the sensors and your brain is the computer
- In its essence, estimation is nothing more than taking noisy sensor data, filtering the noise, and producing the ‘best’ state of the vehicle

What Do We Estimate?

- As NASA engineers, we estimate a variety of things
  - Position, Velocity, Attitude
  - Mass
  - Temperature
  - Sensor parameters (biases)
- These quantities are usually referred to as the ‘states’ of the system
- We use a variety of sensors to accomplish this task
  - Inertial Measurement Units (IMUs)
  - GPS Receivers (GPSRs)
  - LIDARs
  - Cameras
- These sensors are used to determine the states of the system

A Brief History of Estimation

- Estimation has its origins in the work of Gauss and his innovation called 'Least Squares' Estimation
  - He was interested in computing the orbits of asteroids and comets given a set of observations
- Much of the work through WWI centered around extensions to Least Squares Estimation
- In the interval between WWI and WWII, a number of revolutionary contributions were made to sampling and estimation theory
  - Norbert Weiner and the Weiner Filter
  - Claude Shannon and Sampling Theory
- Much of the work in the first half of the Twentieth Century focused on analog circuitry and the frequency domain

Modern Estimation and Rudolf Kalman

- Everything changed with the confluence of two events:
  - The Cold War and the Space Race
  - The Advent of the Digital Computer and Semiconductors
- A new paradigm was introduced: State Space Analysis
  - Linear Systems and Modern Control Theory
    - Estimation Theory
    - Optimization Theory
- Rudolf Kalman proposes a new approach to linear systems
  - Controllability and Observability

Rudolf Kalman and His Filter

- In 1960 Kalman wrote a paper in an obscure ASME journal entitled “A New Approach to Linear Filtering and Prediction Problems” which might have died on the vine, except:
  - In 1961, Stanley Schmidt of NASA Ames read the paper and invited Kalman to give a seminar at Ames
  - Schmidt recognized the importance of this new theory and applied it to the problem of on-board navigation of a lunar vehicle – after all this was the beginning of Apollo
  - This became known as the ‘Kalman Filter’
- Kalman’s paper was rather obtuse in its nomenclature and mathematics
  - It took Schmidt’s exposition to show that this filter could be easily mechanized and applied to a ’real’ problem
- The Kalman Filter became the basis for the on-board navigation filter on the Apollo CSM and LM

Types of Estimation

- There are basically two types of estimation: batch and sequential
- Batch Estimation
  - When sets of measurements taken over a period of time are ‘batched’ and processed together to estimate the state of a vehicle at a given epoch
  - This is usually the case in a ground navigation processor
- Sequential Estimation
  - When measurements are processed as they are taken and the state of the vehicle is updated as the measurements are processed
  - This is done in an on-board navigation system

Types of Sensors

- Inertial Measurement Units (IMUs)
- GPS Recievers
- Magnetometers
- Optical Sensors
  - Visible Cameras
  - IR Cameras
  - LIDARs (Scanning and Flash)
- RF sensors
  - Radars (S-band and C-band)
  - Range and Range-rate from Comm
- Altimeters
- Doppler Velocimeters

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Why do we care about this probability stuff?

"Information: the negative reciprocal value of probability ."

Claude Shannon

Concepts from Probability Theory

- A **random variable** is one whose 'value' is subject to variations due to chance (randomness) – it does not have a fixed 'value'; it can be discrete or continuous
  - A coin toss: can be 'heads' or 'tails' – discrete
  - The lifetime of a light bulb – continuous
- A **probability density function** (pdf), $p(x)$, represents the likelihood that x occurs
  - Always non-negative
  - Satisfies

$$\int_{-\infty}^{\infty} p(\xi)\, d\xi = 1$$

- The **expectation operator**, $E[f(x)]$, is defined as

$$E\left[f(x)\right] = \int_{-\infty}^{\infty} f(\xi)\, p(\xi)\, d\xi$$

Concepts from Probability Theory – Mean and Variance

- The **mean** (or first moment) of a random variable x, denoted by $\bar{x}$, is defined as

$$\bar{x} \triangleq E[x] = \int_{-\infty}^{\infty} \xi \, p(\xi) \, d\xi$$

- The **mean-square** of a random variable x, $E\left[x^2\right]$, is defined as

$$E\left[x^2\right] \triangleq \int_{-\infty}^{\infty} \xi^2 \, p(\xi) \, d\xi$$

- The **variance** (or second moment) of a random variable x, denoted by σ_x^2, is

$$\begin{aligned} \sigma_x^2 \triangleq E\left[[x - E(x)]^2\right] &= \int_{-\infty}^{\infty} (\xi - E(\xi))^2 \, p(\xi) \, d\xi \\ &= E\left[x^2\right] - \bar{x}^2 \end{aligned}$$

Concepts from Probability Theory – Mean and Variance of a Vector

- The **mean** of a random n–vector $\mathbf{x}$, $\bar{\mathbf{x}}$, is defined as

$$\bar{\mathbf{x}} \triangleq E[\mathbf{x}] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi \, p(\xi) \, d\xi$$

- The **(co-)variance** of random n–vector $\mathbf{x}$, $\mathbf{P}_{\mathbf{x}}$, is defined as

$$\mathbf{P}_{\mathbf{x}} \triangleq E\left[[\mathbf{x} - \bar{\mathbf{x}}][\mathbf{x} - \bar{\mathbf{x}}]^T\right] = \int_{-\infty}^{\infty} \left[\xi - \bar{\xi}\right]\left[\xi - \bar{\xi}\right]^T p(\xi) \, d\xi$$

$$= \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} & \cdots & \sigma_{x_1 x_n} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_1 x_n} & \sigma_{x_2 x_n} & \cdots & \sigma_{x_n}^2 \end{bmatrix}$$

The covariance is geometrically represented by an *error ellipsoid*.

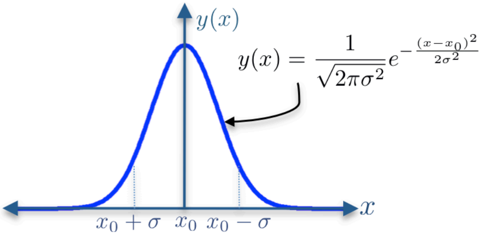
Concepts from Probability Theory –The Gaussian Distribution

- The **Gaussian probability distribution function**, also called the 'Normal distribution'[1] or a 'bell curve', is at the heart of Kalman filtering
- We assume that 'our' random variables have Gaussian pdfs
- We like to work with Gaussians because they are completely characterized by their mean and covariance
  - Linear combinations of Gaussians are Gaussian
- The Gaussian distribution of random n–vector $\mathbf{x}$, with a mean of $\bar{\mathbf{x}}$ and covariance $\mathbf{P_x}$, is defined as

$$p_g(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\mathbf{P_x}|} e^{-\frac{(\mathbf{x}-\bar{\mathbf{x}})^T \mathbf{P_x}^{-1} (\mathbf{x}-\bar{\mathbf{x}})}{2}}$$

[1] Physicist G. Lippman is reported to have said, 'Everyone believes in the normal approximation, the experimenters because they think it is a mathematical theorem, the mathematicians because they think it is an experimental fact.'

Concepts from Probability Theory –The Gaussian Distribution



- We can show that

$$\int_{\mathcal{R}^n} \frac{1}{(2\pi)^{n/2} |\mathbf{P_x}|} e^{-\frac{(\mathbf{x}-\bar{\mathbf{x}})^T \mathbf{P_x}^{-1} (\mathbf{x}-\bar{\mathbf{x}})}{2}} d\mathbf{x} = 1$$

- If a random process is generated by a sum of other (non-Gaussian) random processes, then, in the limit, the combined distribution approaches a Gaussian distribution (*The Central Limit Theorem*)

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Linear Systems

- A **system** is a mapping from input signals to output signals, written as: $w(t) = L(v(t))$
- A system is **linear** if for all input signals $v(t)$, $v_1(t)$, and $v_2(t)$ and for all scalars α,
  - L is *additive*: $L(v_1(t) + v_2(t)) = L(v_1(t)) + L(v_2(t))$
  - L is *homogeneous*: $L(\alpha v(t)) = \alpha L(v(t))$
- For a system to be linear, if 0 is an input, then 0 is an output:
$$L(0) = L(0 \cdot v(t)) = 0 \cdot L(v(t)) = 0$$
- If the system does not satisfy the above two properties, it is said to be **nonlinear**
- If $L(v(t)) = v(t) + 1$, is this linear?
  - It is not because for $v(t) = 0$, $L(0) = 1 \neq 0$
- Lesson: Some systems may *look* linear but they are not!

Nonlinear Systems and the Linearization Process

- Despite the beauty associated with linear systems, the fact of the matter is that we live in a nonlinear world.
- So, what do we do? We make these nonlinear systems into linear systems by **linearizing**
- This is predicated on a Taylor series approximation which we deploy as follows: Given a nonlinear system of the form: $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, t)$, we linearize about (or expand about) a nominal trajectory, $\mathbf{X}^\star$ (with $\dot{\mathbf{X}}^\star = \mathbf{f}(\mathbf{X}^\star, t)$), as

$$\dot{\mathbf{X}}(t) = f(\mathbf{X}^\star, t) + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right)_{\mathbf{X}=\mathbf{X}^\star} (\mathbf{X} - \mathbf{X}^\star) + \cdots$$

Nonlinear Systems and the State Transition Matrix

- If we let $\mathbf{x}(t) = \mathbf{X} - \mathbf{X}^\star$ and let $\mathbf{F}(t) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right)_{\mathbf{X}=\mathbf{X}^\star}$, then we get

$$\dot{\mathbf{x}} = \mathbf{F}(t)\mathbf{x} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

- The solution of this equation is

$$\mathbf{x}(t) = e^{\int_{t_0}^{t} \mathbf{F}(\tau)\, d\tau}\mathbf{x}_0 = \mathbf{\Phi}(t, t_0)\mathbf{x}_0$$

where $\mathbf{\Phi}(t, t_0)$ is the **State Transition Matrix** (STM) which satisfies

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{F}(t)\mathbf{\Phi}(t, t_0) \quad \text{with} \quad \mathbf{\Phi}(t_0, t_0) = \mathbf{I}$$

- The STM can be approximated (for $\mathbf{F}(t) = \mathbf{F} =$ a constant) as

$$\mathbf{\Phi}(t, t_0) = e^{\int_{t_0}^{t} \mathbf{F}(\tau)\, d\tau} = e^{\mathbf{F}(t-t_0)} = \mathbf{I} + \mathbf{F}(t - t_0) + \frac{1}{2}\mathbf{F}^2(t - t_0)^2 + \cdots$$

A Bit More About the State Transition Matrix

The State Transition Matrix (STM) is at the heart of practical Kalman filtering. In its essence it is defined as

$$\mathbf{\Phi}(t, t_0) \triangleq \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)}$$

As the name implies, it is used to 'transition' or **move** perturbations of the state of a nonlinear system from one epoch to another, *i.e.*

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) \iff \left(\mathbf{X}(t) - \mathbf{X}^\star(t)\right) = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)}\left(\mathbf{X}(t_0) - \mathbf{X}^\star(t_0)\right)$$

In practical Kalman filtering, we use a first-order approximation[2]

$$\mathbf{\Phi}(t, t_0) \approx \mathbf{I} + \mathbf{F}(t_0)\,(t - t_0) = \mathbf{I} + \left.\frac{\partial \mathbf{f}(\mathbf{X}, t)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_0} (t - t_0)$$

[2] In cases of fast dynamics, we can approximate the STM to second-order as:
$$\mathbf{\Phi}(t, t_0) \approx \mathbf{I} + \mathbf{F}(t_0)\,(t - t_0) + \tfrac{1}{2}\mathbf{F}^2(t_0)\,(t - t_0)^2$$

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How do we Implement This?

"Never do a calculation unless you already know the answer."

John Archibald Wheeler's First Moral Principle

The Context of Least Squares Estimation

- Least Squares estimation has been a mainstay of engineering and science since Gauss invented it to track Ceres circa 1794
- It has been used extensively for spacecraft state estimation, particularly in **ground-based navigation systems**
- The Apollo program had an extensive ground station network (MSFN/STDN) coupled with sophisticated ground-based batch processors for tracking the CSM and LM
  - A set of measurements (or several sets of measurements) taken over many minutes and over several passes from different ground stations would be 'batched' together to get a spacecraft state at a particular epoch
- Least Squares estimation is predicated on finding a solution which minimizes the square of the errors of the model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\epsilon}$$

The Least Squares Problem

The problem is as follows: given a set of observations, $\mathbf{y}$, subject to measurement errors ($\boldsymbol{\epsilon}$), find the best solution, $\hat{\mathbf{x}}$, which minimizes the errors, i.e.

$$\min J = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)^T\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)$$

To do this we take the first derivative of J with respect to $\mathbf{x}$ and set it equal to zero as

$$\frac{\partial J}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left[\frac{1}{2}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)^T\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)\right]_{\mathbf{x}=\hat{\mathbf{x}}} = -\left(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\right)^T\mathbf{H} = 0$$

Therefore, the optimal solution, $\hat{\mathbf{x}}$, is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$$

The Weighted Least Squares (WLS) Problem

Suppose now we are given measurements $\mathbf{y}$, whose error has a measurement covariance of $\mathbf{R}$. How can we get the best estimate, $\hat{\mathbf{x}}$ which minmizes the errors weighted by the accuracy of the measurement error ($\mathbf{R}^{-1}$)? The problem can be posed as

$$\min J = \frac{1}{2}\boldsymbol{\epsilon}^T\mathbf{R}^{-1}\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)^T \mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)$$

Once again, we take the first derivative of J with respect to $\mathbf{x}$ and set it equal to zero as

$$\frac{\partial J}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left[\frac{1}{2}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)^T \mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)\right]_{\mathbf{x}=\hat{\mathbf{x}}} = -\left(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\right)^T \mathbf{R}^{-1}\mathbf{H} = 0$$

Therefore, the optimal solution, $\hat{\mathbf{x}}$, is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}$$

The WLS Problem with *A Priori* Information

Suppose we need to find the best estimate of the state, given measurements $\mathbf{y}$, with measurement error covariance $\mathbf{R}$, but we are also given an *a priori* estimate of the state, $\bar{\mathbf{x}}$ with covariance $\bar{\mathbf{P}}$. This problem can be posed as

$$\min J = \frac{1}{2}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right)^{T}\mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{x}\right) + \frac{1}{2}\left(\bar{\mathbf{x}} - \mathbf{x}\right)^{T}\bar{\mathbf{P}}^{-1}\left(\bar{\mathbf{x}} - \mathbf{x}\right)$$

As before, we take the first derivative of J with respect to $\mathbf{x}$ and set it equal to zero as

$$\left.\frac{\partial J}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}} = -\left(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\right)^{T}\mathbf{R}^{-1}\mathbf{H} - \left(\bar{\mathbf{x}} - \hat{\mathbf{x}}\right)^{T}\bar{\mathbf{P}}^{-1} = 0$$

Therefore, the optimal solution, $\hat{\mathbf{x}}$, is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{H} + \bar{\mathbf{P}}^{-1}\right)^{-1}\left[\mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{y} + \bar{\mathbf{P}}^{-1}\bar{\mathbf{x}}\right]$$

Nonlinear Batch Estimation

In general, the system of interest will be nonlinear of the form

$$\mathbf{Y}_k = \mathbf{h}(\mathbf{X}_k, t_k) + \boldsymbol{\epsilon}_k$$

How do we get the best estimate of the state $\mathbf{X}$? Well, first we linearize about a nominal state $\mathbf{X}_k^\star$ (with $\mathbf{x}_k \triangleq \mathbf{X}_k - \mathbf{X}_k^\star$) as

$$\mathbf{Y}_k = \mathbf{h}(\mathbf{X}_k^\star + \mathbf{x}_k, t_k) + \boldsymbol{\epsilon}_k = \mathbf{h}(\mathbf{X}_k^\star) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{X}}\right|_{\mathbf{X}_k = \mathbf{X}_k^\star} \left(\mathbf{X}_k - \mathbf{X}_k^\star\right) + \cdots + \boldsymbol{\epsilon}_k$$

Defining $\tilde{\mathbf{H}}_k \triangleq \left.\frac{\partial \mathbf{h}}{\partial \mathbf{X}}\right|_{\mathbf{X}_k = \mathbf{X}_k^\star}$ we get the following equation

$$\mathbf{y}_k = \tilde{\mathbf{H}}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k$$

Nonlinear Batch Estimation at an Epoch

In batch estimation, we are interested in estimating a state at an epoch, say $\mathbf{X}_0$, with measurements taken after that epoch – say, at t_k. How can we obtain this? Well, we use the state transition matrix as follows

$$\mathbf{X}_k - \mathbf{X}_k^\star = \mathbf{\Phi}(t_k, t_0)\left(\mathbf{X}_k - \mathbf{X}_k^\star\right) \iff \mathbf{x}_k = \mathbf{\Phi}(t_k, t_0)\mathbf{x}_0$$

so that we can map the measurements back to the epoch of interest as

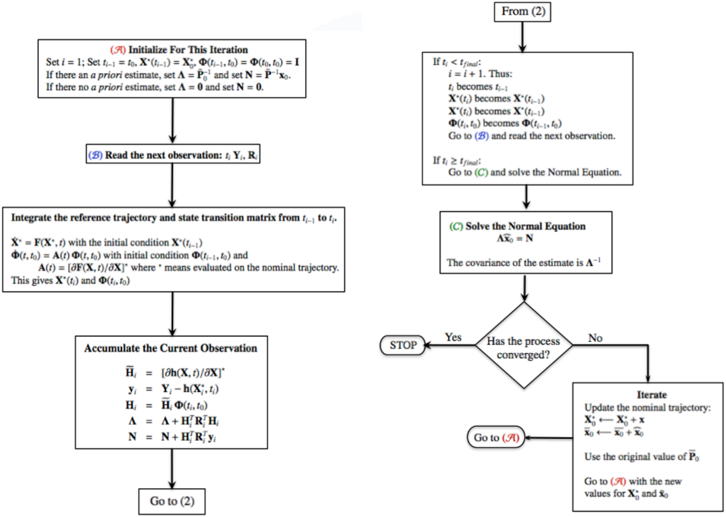
$$\mathbf{y}_k = \tilde{\mathbf{H}}_k \mathbf{\Phi}(t_k, t_0)\mathbf{x}_0 + \boldsymbol{\epsilon}_k = \mathbf{H}_k \mathbf{x}_0 + \boldsymbol{\epsilon}_k$$

The least squares solution (over all the p measurements) is

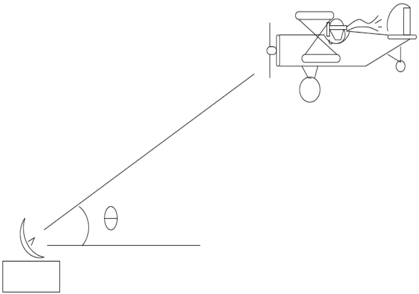
$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{p} \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i + \bar{\mathbf{P}}_0^{-1}\right)^{-1} \left[\sum_{i=1}^{p} \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{y}_i + \bar{\mathbf{P}}_0^{-1} \bar{\mathbf{x}}_0\right] = \hat{\mathbf{X}}_0 - \mathbf{X}_0^\star$$

This is called the **normal equation**.

The Nonlinear Batch Estimation Algorithm



Batch Filter Example – Aircraft Tracking



Given a ground station tracking an airplane, moving in a straight line at a constant speed, with only bearing measurements, we are interested in knowing the speed of the airplane and its position at the beginning of the tracking pass (x_0, y_0, u_0, v_0). The equations are

$$\begin{aligned} x(t) &= u_0(t - t_0) + x_0 \\ y(t) &= v_0(t - t_0) + y_0 \\ \theta(t) &= \tan^{-1}\left[\frac{y(t)}{x(t)}\right] \end{aligned}$$

Batch Filter Example – Aircraft Tracking (II)

The measurements are:

k	t_k	θ_k (degrees)
0	0	5.4628
1	20	18.9309
2	40	33.4603
3	60	45.1648
4	80	53.7033
5	100	62.3816
6	120	68.1143
7	140	71.9306
8	160	75.7515
9	180	78.5952
10	200	80.8027

The initial guess is

$$\mathbf{X}_0^* = \begin{bmatrix} 985 \\ 105 \\ -1.5 \\ 10 \end{bmatrix}$$

with initial covariance

$$\mathbf{P}_0 = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Batch Filter Example – Aircraft Tracking (III)

After 7 iterations the following results are obtained:

Parameter	Truth	Initial Guess	Converged State
x_0	1000	985	983.5336
y_0	100	105	99.3470
u_0	-3	-1.5	-2.9564
v_0	12	10	11.7763

Lesson: The x–component is not readily observable. But that is not surprising since angles do not provide information along the line-of-sight.

Something to remember

One must watch the convergence of a numerical code as carefully as a father watching his four year old play near a busy road.

J. P. Boyd

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The Need for Careful Preparation

"Six months in the lab can save you a day in the library"

Albert Migliori, quoted by J. Maynard
in *Physics Today 49*, 27 (1996)

Stochastic Processes – The Linear First-Order Differential Equation

- Let us look at a first-order differential equation for $x(t)$, given $f(t)$, $g(t)$, $w(t)$ and x_0 as

$$\dot{x}(t) = f(t)x(t) + g(t)w(t) \quad \text{with} \quad x(t_0) = x_0$$

- The solution of this equation is

$$x(t) = e^{\int_{t_0}^{t} f(\tau)d\tau} x_0 + \int_{t_0}^{t} e^{\int_{\xi}^{t} f(\tau)d\tau} g(\xi)w(\xi)d\xi$$

- Suppose now we define $\phi(t, t_0) \triangleq e^{\int_{t_0}^{t} f(\tau)d\tau}$, we can write the above solution as

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \xi)g(\xi)w(\xi)d\xi$$

The Mean of a Linear First-Order Stochastic Process

- Given a first-order *stochastic process*, $\chi(t)$, with constant f and g and white noise, $w(t)$, which is represented as

$$\dot{\chi}(t) = f\,\chi(t) + g\,w(t) \quad \text{with} \quad \chi(t_0) = \chi_0$$

and the mean and covariance of $w(t)$ expressed as

$$E[w(t)] = 0 \quad \text{and} \quad E[w(t)w(\tau)] = q\,\delta(t-\tau)$$

- The mean of the process, $\bar{\chi}(t)$ is

$$\begin{aligned}
\bar{\chi}(t) = E[\chi(t)] &= e^{\int_{t_0}^{t} f\,d\tau}\bar{\chi}_0 + \int_{t_0}^{t} e^{\int_{\xi}^{t} f\,d\tau} g(\xi)E[w(\xi)]d\xi \\
&= e^{\int_{t_0}^{t} f\,d\tau}\bar{\chi}_0 \\
&= e^{f(t-t_0)}\bar{\chi}_0
\end{aligned}$$

Stochastic Processes – The Mean-Square and Covariance of a Linear First-Order Stochastic Process

- The mean-square of the linear first-order stochastic process, $\chi(t)$ is

$$\begin{aligned} E[\chi^2(t)] &= e^{2f(t-t_0)} E\left[\chi(t_0)\chi(t_0)\right] + \frac{q}{2f}\left[1 - e^{2f(t-t_0)}\right] \\ &= \phi^2(t,t_0) E\left[\chi(t_0)\chi(t_0)\right] + \frac{q}{2f}\left[1 - \phi^2(t,t_0)\right] \end{aligned}$$

- The covariance of $\chi(t)$, $P_{\chi\chi}(t)$, is expressed as

$$\begin{aligned} P_{\chi\chi}(t) &= E\left[(\chi(t) - \bar{\chi}(t))^2\right] = E[\chi^2(t)] - \bar{\chi}^2(t) \\ &= \phi^2(t,t_0) P_{\chi\chi}(t_0) + \frac{q}{2f}\left[1 - \phi^2(t,t_0)\right] \end{aligned}$$

Stochastic Processes – The Vector First-Order Differential Equation

A first-order **vector** differential equation for $\mathbf{x}(t)$, given $\mathbf{x}(t_0)$ and white noise with $E(\mathbf{w}(t)) = \mathbf{0}$, and $E(\mathbf{w}(t)\mathbf{w}(\tau^T)) = \mathbf{Q}\,\delta(t-\tau)$, is

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t)$$

The solution of this equation is

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t,\xi)\mathbf{G}(\xi)\mathbf{w}(\xi)d\xi$$

where $\mathbf{\Phi}(t,t_0)$ satisfies the following equation

$$\dot{\mathbf{\Phi}}(t,t_0) = \mathbf{F}(t)\mathbf{\Phi}(t,t_0), \quad \text{with} \quad \mathbf{\Phi}(t_0,t_0) = \mathbf{I}$$

The Mean and Mean-Square of a Linear, Vector Process

The mean of the stochastic process $\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t)$ is

$$\begin{aligned}\bar{\mathbf{x}}(t) &= E[\mathbf{x}(t)] = \mathbf{\Phi}(t, t_0)E[\mathbf{x}(t_0)] + \int_{t_0}^{t} \mathbf{\Phi}(t, \xi)\mathbf{G}(\xi)E[\mathbf{w}(\xi)]d\xi \\ &= \mathbf{\Phi}(t, t_0)\bar{\mathbf{x}}(t_0)\end{aligned}$$

The mean-square of the process (with $E[\mathbf{x}(t_0)\mathbf{w}^T(t)] = \mathbf{0}$) is

$$\begin{aligned}E[\mathbf{x}(t)\mathbf{x}^T(t)] &= E\left\{\left[\mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \xi)\mathbf{G}(\xi)\mathbf{w}(\xi)d\xi\right]\right. \\ &\quad \left.\times\left[\mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \chi)\mathbf{G}(\xi)\mathbf{w}(\chi)d\chi\right]\right\} \\ &= \mathbf{\Phi}(t, t_0)E[\mathbf{x}(t_0)\mathbf{x}^T(t_0)]\mathbf{\Phi}^T(t, t_0) \\ &\quad + \int_{t_0}^{t} \mathbf{\Phi}(t, \xi)\mathbf{G}(\xi)\,\mathbf{Q}\,\mathbf{G}^T(\xi)\mathbf{\Phi}^T(t, \xi)\,d\xi\end{aligned}$$

The Covariance of a Linear, Vector Process

The covariance of $\mathbf{x}(t)$, $\mathbf{P_{xx}}(t)$, given $\mathbf{P_{xx}}(t_0)$, is expressed as

$$\begin{aligned}\mathbf{P_{xx}}(t) &= E\left[(\mathbf{x}(t)-\bar{\mathbf{x}}(t))\,(\mathbf{x}(t)-\bar{\mathbf{x}}(t))^T\right] = E[\mathbf{x}(t)\mathbf{x}^T(t)] - \bar{\mathbf{x}}(t)\bar{\mathbf{x}}^T(t) \\ &= \mathbf{\Phi}(t,t_0)\mathbf{P_{xx}}(t_0)\mathbf{\Phi}^T(t,t_0) \\ &\quad + \int_{t_0}^{t} \mathbf{\Phi}(t,\xi)\mathbf{G}(\xi)\,\mathbf{Q}\,\mathbf{G}^T(\xi)\mathbf{\Phi}^T(t,\xi)\,d\xi\end{aligned}$$

The differential equation for $\mathbf{P_{xx}}(t)$ can be found to be

$$\dot{\mathbf{P}}_{\mathbf{xx}}(t) = \mathbf{F}(t)\mathbf{P_{xx}}(t) + \mathbf{P_{xx}}(t)\mathbf{F}^T(t) + \mathbf{G}(t)\,\mathbf{Q}\,\mathbf{G}^T(t)$$

In the above development we have made use of *the Sifting Property of the Dirac Delta*, $\delta(t-\tau)$, expressed as

$$\int_{-\infty}^{\infty} f(\xi)\delta(t-\xi)d\xi = f(t)$$

A Discrete Linear, Vector Process

Given the continuous process ($\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t)$), whose solution is

$$\mathbf{x}(t_k) = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t,\xi)\mathbf{G}(\xi)\mathbf{w}(\xi)d\xi$$

the discrete stochastic analog process is

$$\mathbf{x}_k = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{x}_{k-1} + \mathbf{w}_k, \quad \text{with} \;\; \mathbf{w}_k \stackrel{\Delta}{=} \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t,\xi)\mathbf{G}(\xi)\mathbf{w}(\xi)d\xi$$

whose mean is

$$\bar{\mathbf{x}}_k = \mathbf{\Phi}(t_k, t_{k-1})\bar{\mathbf{x}}_{k-1}$$

The Covariance of a Discrete Linear, Vector Process

Likewise, the continuous-time solution for the covariance was

$$\begin{aligned}\mathbf{P_{xx}}(t_k) &= \mathbf{\Phi}(t_k, t_0)\mathbf{P_{xx}}(t_0)\mathbf{\Phi}^T(t_k, t_0) \\ &\quad + \int_{t_0}^{t} \mathbf{\Phi}(t_k, \xi)\mathbf{G}(\xi)\,\mathbf{Q}\,\mathbf{G}^T(\xi)\mathbf{\Phi}^T(t_k, \xi)\, d\xi\end{aligned}$$

whose discrete analog is

$$\mathbf{P_{xx}}_k = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{P_{xx}}_{k-1}\mathbf{\Phi}^T(t_k, t_{k-1}) + \mathbf{Q}_k$$

where

$$\mathbf{Q}_k \triangleq \int_{t_0}^{t} \mathbf{\Phi}(t_k, \xi)\mathbf{G}(\xi)\,\mathbf{Q}\,\mathbf{G}^T(\xi)\mathbf{\Phi}^T(t_k, \xi)\, d\xi$$

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“There is nothing more practical than a good theory”

Albert Einstein

The Context of the Kalman Filter

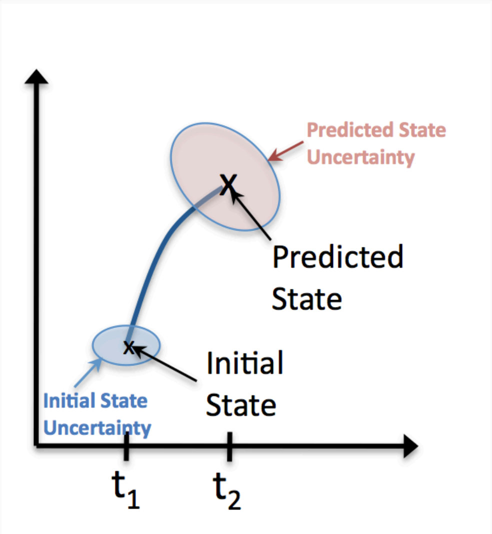
- With the advent of the digital computer and modern control, the following question arose: Can we recursively estimate the state of a vehicle as measurements become available?
- In 1961 Rudolf Kalman came up with just such a methodology to compute an optimal state given linear measurements and a linear system
- The resulting *Kalman filter* is an globally optimal linear, model-based estimator driven by Gaussian, white noise which has two steps
    - Propagation: the state and covariance are propagated from one epoch to the next by integrating model-based dynamics
    - Update: the state and covariance are optimally updated with measurements
- We begin with the same equation as before

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k \quad \text{with} \quad E(\boldsymbol{\epsilon}_k) = \mathbf{0},\ E(\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^T) = \mathbf{R}_k$$

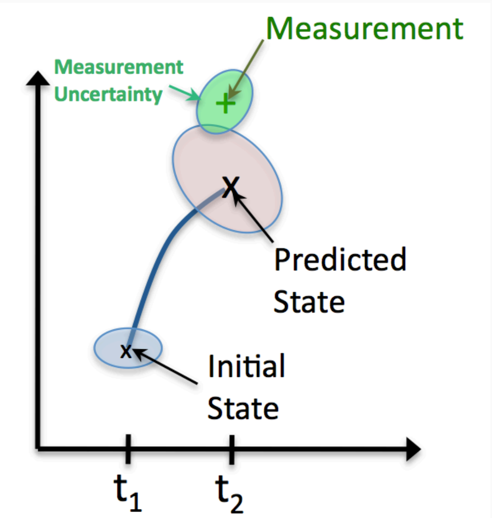
What does a Kalman Filter do?

- Fundamentally, a Kalman filter is nothing more than a predictor (which we call the ‘propagation’ phase) followed by a corrector (which we call the ‘update’ phase)
- We use the dynamics (*i.e.* Newton’s Laws) to *predict* the state at the time of a measurement
- The measurements are then used to *correct* or update the predicted state.
- It does this in an “optimal” fashion

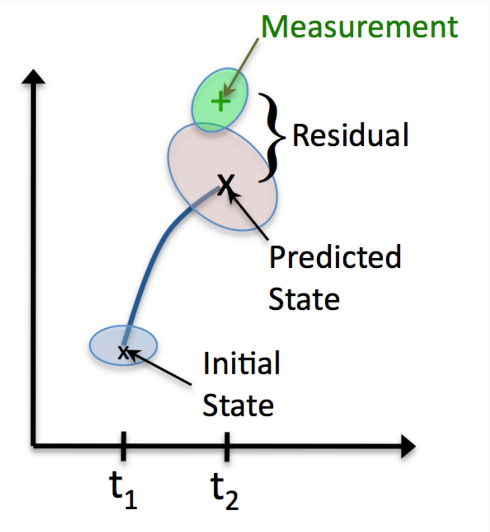
Prediction


Predicted State
Uncertainty
Predicted
State
Initial
State
Initial State
Uncertainty
t1
t2

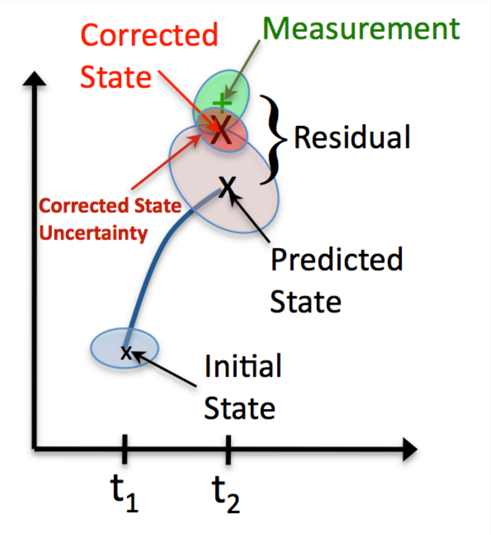
Measurement



Compute Residual



Correction


Corrected State
Measurement
Residual
Corrected State Uncertainty
Predicted State
Initial State
t1
t2

The Derivation of the Kalman Filter (I)

Let $\hat{\mathbf{x}}_k^-$ be an unbiased *a priori* estimate (the **prediction**) of $\mathbf{x}_k$ with covariance $\mathbf{P}_k^-$ so that the *a priori* estimate error, $\mathbf{e}_k^-$ is

$$\mathbf{e}_k^- = \mathbf{x}_k - \hat{\mathbf{x}}_k^- \quad \text{with} \quad E(\mathbf{e}_k^-) = \mathbf{0},\ E(\mathbf{e}_k^- \mathbf{e}_k^{-T}) = \mathbf{P}_k^-$$

We hypothesize an unbiased linear update (the **correction**) to $\mathbf{x}_k$, called $\hat{\mathbf{x}}_k^+$, as follows (with $\mathbf{K}_k$ as yet unknown)

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^- \right)$$

whose *a posteriori* error, $\mathbf{e}_k^+$, is

$$\mathbf{e}_k^+ = \mathbf{x}_k - \hat{\mathbf{x}}_k^+ = \mathbf{e}_k^- - \mathbf{K}_k(\mathbf{H}_k \mathbf{e}_k^- + \epsilon_k) = (\mathbf{I}_k - \mathbf{K}_k \mathbf{H}_k)\mathbf{e}_k^- - \mathbf{K}_k \epsilon$$

If $\mathbf{e}_k^-$ and ϵ_k are uncorrelated, then the *a posteriori* covariance is

$$\mathbf{P}_k^+ = E(\mathbf{e}_k^+ \mathbf{e}_k^{+T}) = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T$$

The Derivation of the Kalman Filter (II)

So far we haven't said anything about $\mathbf{K}_k$. We now choose $\mathbf{K}_k$ to minimize the *a posteriori* error as[1]

$$\begin{aligned}\min J &= \frac{1}{2}E\left[\mathbf{e}_k^{+T}\mathbf{e}_k^{+}\right] = \frac{1}{2}\mathrm{tr}\left\{E\left[\mathbf{e}_k^{+T}\mathbf{e}_k^{+}\right]\right\} = \frac{1}{2}E\left\{\mathrm{tr}\left[\mathbf{e}_k^{+T}\mathbf{e}_k^{+}\right]\right\} \\ &= \frac{1}{2}E\left\{\mathrm{tr}\left[\mathbf{e}_k^{+}\mathbf{e}_k^{+T}\right]\right\} = \frac{1}{2}\mathrm{tr}\left\{E\left[\mathbf{e}_k^{+}\mathbf{e}_k^{+T}\right]\right\} = \frac{1}{2}\mathrm{tr}\left(\mathbf{P}_k^{+}\right)\end{aligned}$$

so we obtain $\mathbf{K}$ by[2]

$$\frac{\partial}{\partial \mathbf{K}_k}\mathrm{tr}\left(\mathbf{P}_k^{+}\right) = \frac{\partial}{\partial \mathbf{K}_k}\mathrm{tr}\left[(\mathbf{I}-\mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^{-}(\mathbf{I}-\mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T\right] = \mathbf{0}$$

[1] The *cyclic invariance* property of the trace is: $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{BCA}) = \mathrm{tr}(\mathbf{CAB})$

[2] Recalling that

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{AXBX}^T) = \mathbf{A}^T\mathbf{XB}^T + \mathbf{AXB};\quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{AXB}) = \mathbf{A}^T\mathbf{B}^T;\quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{AX}^T\mathbf{B}) = \mathbf{BA}$$

The Derivation of the Kalman Filter (III)

This results in the following condition

$$-\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{K}_{k_{opt}}\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k\right)^T + \mathbf{K}_{k_{opt}}\left(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^T + \mathbf{R}_k\right) = \mathbf{0}$$

which gives

$$\mathbf{K}_{k_{opt}} = \mathbf{P}_k^-\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}$$

and substituting into the equation[2] for $\mathbf{P}^+$ we get

$$\mathbf{P}_k^+ = \mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}\mathbf{H}_k\mathbf{P}_k^- = \left(\mathbf{I} - \mathbf{K}_{k_{opt}}\mathbf{H}_k\right)\mathbf{P}_k^-$$

so the state update is

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_{k_{opt}}\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-\right)$$

[2] Recall that $\mathbf{P}^+ = (\mathbf{I} - \mathbf{KH})\mathbf{P}^-(\mathbf{I} - \mathbf{KH})^T + \mathbf{KRK}^T$

The Kalman Filter Revealed

Given the dynamics and the measurements

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{\Phi}(t_k, t_{k-1})\mathbf{x}_{k-1} + \mathbf{\Gamma}_k \mathbf{w}_k, \quad \text{with } E(\mathbf{w}_k) = \mathbf{0},\ E(\mathbf{w}_k \mathbf{w}_j^T) = \mathbf{Q}_k \delta_{kj} \\
\mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k, \quad \text{with } E(\boldsymbol{\epsilon}_k) = \mathbf{0},\ E(\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_j^T) = \mathbf{R}_k \delta_{kj}
\end{aligned}
$$

The Kalman Filter contains the following phases:

Propagation – the Covariance Increases

$$
\begin{aligned}
\hat{\mathbf{x}}_k^- &= \mathbf{\Phi}(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1}^+ \\
\mathbf{P}_k^- &= \mathbf{\Phi}(t_k, t_{k-1})\mathbf{P}_{k-1}^+ \mathbf{\Phi}^T(t_k, t_{k-1}) + \mathbf{\Gamma}_k \mathbf{Q}_k \mathbf{\Gamma}_k^T
\end{aligned}
$$

Update – the Covariance Decreases

$$
\begin{aligned}
\mathbf{K}_{k_{opt}} &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\hat{\mathbf{x}}_k^+ &= \hat{\mathbf{x}}_k^- + \mathbf{K}_{k_{opt}} \left(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^- \right) \\
\mathbf{P}_k^+ &= \left(\mathbf{I} - \mathbf{K}_{k_{opt}} \mathbf{H}_k \right) \mathbf{P}_k^- = \left(\mathbf{I} - \mathbf{K}_{k_{opt}} \mathbf{H}_k \right) \mathbf{P}_k^- \left(\mathbf{I} - \mathbf{K}_{k_{opt}} \mathbf{H}_k \right)^T + \mathbf{K}_{k_{opt}} \mathbf{R}_k \mathbf{K}_{k_{opt}}^T
\end{aligned}
$$

A Kalman Filter Example

Given a spring-mass-damper system governed by the following equation

$$\ddot{r}(t) = -0.001 r(t) - 0.005 \dot{r}(t) + w(t)$$

the system can be written (in first-order discrete form, $\mathbf{x}_k = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{x}_{k-1} + \mathbf{\Gamma}_k \mathbf{w}_k$) as

$$\begin{bmatrix} r(t_k) \\ \dot{r}(t_k) \end{bmatrix} = \exp\left\{ \begin{bmatrix} 0 & 1 \\ -0.001 & -0.005 \end{bmatrix} \Delta t \right\} \begin{bmatrix} r(t_{k-1}) \\ \dot{r}(t_{k-1}) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$

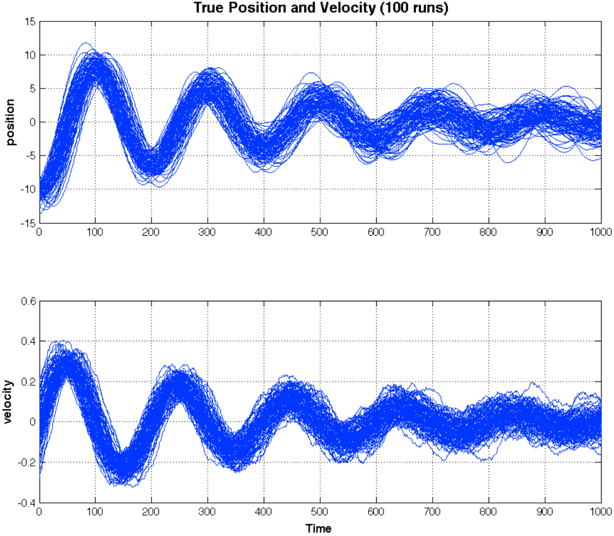
with measurements

$$y_k = r(t_k) + \epsilon_k \quad \text{with} \quad E[\epsilon_k] = 0, \quad E[\epsilon_j \epsilon_k] = 0.001^2 \delta_{jk}$$

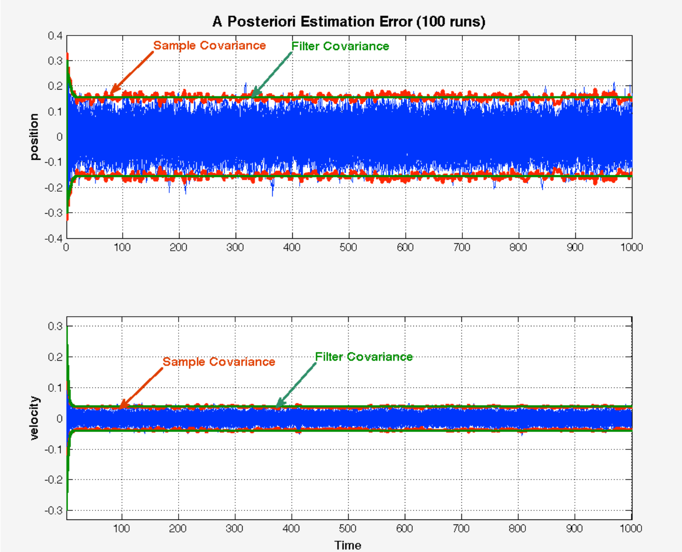
and

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1^2 \end{bmatrix} \quad \text{and} \quad Q = 0.005^2$$

A Kalman Filter Example (II)



A Kalman Filter Example (III)



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"A computation is a temptation that should be resisted a long as possible "

John Boyd (paraphrasing T.S. Eliot) , 2000

The Extended Kalman Filter

Since we live in a nonlinear and non-Gaussian world, can we fit the Kalman filter paradigm into the 'real' world? Being engineers, when all else fails, we linearize.

$$\widehat{\mathbf{X}}_k = \mathbf{X}_k^\star + \hat{\mathbf{x}}_k$$

This process results in an algorithm called *the Extended* Kalman filter (EKF). However all guarantees of stability and optimality are lost. *The EKF is a conditional mean estimator with dynamics truncated after first-order by deterministically linearizing about the conditional mean.*

The Development of the Extended Kalman Filter (I)

Begin with the nonlinear state equation

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\mathbf{X}, t) + \mathbf{w}(t) \quad \text{with} \quad E[\mathbf{w}(t)] = \mathbf{0},\ E[\mathbf{w}(t)\mathbf{w}(\tau)] = \mathbf{Q}\,\delta(t-\tau)$$

whose solution, given $\mathbf{X}(t_{k-1})$ is

$$\mathbf{X}(t) = \mathbf{X}(t_{k-1}) + \int_{t_{k-1}}^{t} \mathbf{f}(\mathbf{X}, \xi)d\xi + \int_{t_{k-1}}^{t} \mathbf{w}(\xi)d\xi$$

We expand $\mathbf{f}(\mathbf{X}, t)$ in a Taylor series about $\hat{\mathbf{X}} = E(\mathbf{X})$ as

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\hat{\mathbf{X}}, t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\hat{\mathbf{X}}} \left(\mathbf{X} - \hat{\mathbf{X}}\right) + \cdots + \mathbf{w}(t)$$

so that $\dot{\hat{\mathbf{X}}}(t)$, neglecting higher than first-order terms,

$$\dot{\hat{\mathbf{X}}}(t) = \mathbf{f}(\hat{\mathbf{X}}, t)$$

The Development of the Extended Kalman Filter (II)

Recalling the definition of $\mathbf{P} \left(\stackrel{\Delta}{=} E\left[(\mathbf{X} - \hat{\mathbf{X}})(\mathbf{X} - \hat{\mathbf{X}})^T \right] \right)$, we find that

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{Q} \quad \text{where} \quad \mathbf{F} \stackrel{\Delta}{=} \left. \frac{\partial \mathbf{f}}{\partial \mathbf{X}} \right|_{\mathbf{X}=\hat{\mathbf{X}}}$$

which can be integrated as

$$\mathbf{P}(t_k) = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{P}(t_{k-1})\mathbf{\Phi}^T(t_k, t_{k-1}) + \mathbf{Q}_k$$

with $\mathbf{\Phi}(t_{k-1}, t_{k-1}) = \mathbf{I}$ and

$$\dot{\mathbf{\Phi}}(t, t_{k-1}) = \mathbf{F}(t)\mathbf{\Phi}(t, t_{k-1}), \quad \text{and} \quad \mathbf{Q}_k = \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k, \xi)\, \mathbf{Q}\, \mathbf{\Phi}^T(t_k, \xi) d\xi$$

The Development of the Extended Kalman Filter (III)

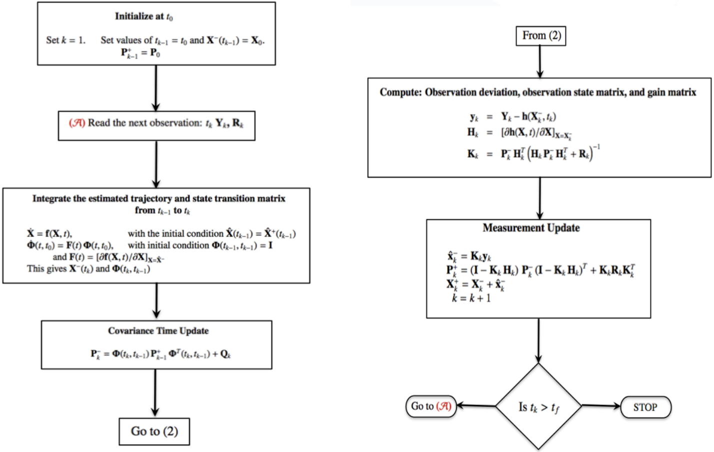
Likewise, the measurement equation can be expanded in a Taylor series about $\hat{\mathbf{X}}_k^-$, the *a priori* state, as

$$\mathbf{Y}_k = \mathbf{h}(\mathbf{X}_k) + \epsilon_k = \mathbf{h}(\hat{\mathbf{X}}_k^-) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{X}_k}\right|_{\mathbf{X}_k = \hat{\mathbf{X}}_k^-} \left(\mathbf{X}_k - \hat{\mathbf{X}}_k^-\right) + \cdots + \epsilon_k$$

In the EKF development, we truncate the Taylor series after first-order. As in the Kalman filter development, we minimize the trace of the *a posteriori* covariance and this results in

$$\begin{aligned}
\mathbf{K}_k(\hat{\mathbf{X}}_k^-) &= \mathbf{P}_k^- \mathbf{H}_k^T(\hat{\mathbf{X}}_k^-) \left[\mathbf{H}_k(\hat{\mathbf{X}}_k^-) \mathbf{P}_k^- \mathbf{H}_k^T(\hat{\mathbf{X}}_k^-) + \mathbf{R}_k\right]^{-1} \\
\mathbf{P}_k^+ &= \left[\mathbf{I} - \mathbf{K}_k(\hat{\mathbf{X}}_k^-)\mathbf{H}_k^T(\hat{\mathbf{X}}_k^-)\right] \mathbf{P}_k^- \\
\hat{\mathbf{X}}_k^+ &= \hat{\mathbf{X}}_k^- + \mathbf{K}_k(\hat{\mathbf{X}}_k^-) \left[\mathbf{Y}_k - \mathbf{h}_k(\hat{\mathbf{X}}_k^-)\right] \\
\mathbf{H}_k(\hat{\mathbf{X}}_k^-) &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{X}_k}\right|_{\mathbf{X}_k = \hat{\mathbf{X}}_k^-}
\end{aligned}$$

The Extended Kalman Filter (EKF) Algorithm



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"In theory, there is no difference between theory and practice, but in practice there is"

John Junkins, 2012

Practical Matters – Processing Multiple Measurements

- In general, more than one measurement will arrive at the same time
- Usually the measurements are uncorrelated and hence they can be processed one-at-a-time
  - However, even if they are correlated, they can usually be treated as if they were uncorrelated – by increasing the measurement noise variance
- If the measurements are processed one-at-a-time, then

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} = \frac{\mathbf{P}_k^- \mathbf{H}_k^T}{\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + R_k}$$

- Thus there is no need for a matrix inverse – we can use scalar division
- This greatly reduces the computational throughput, not to mention software complexity

Practical Matters – Processing Non-Gaussian Measurements

- The Kalman Filter is predicated on measurements whose errors are Gaussian
- However, real-world sensors seldom have error characteristics that are Gaussian
  - Real sensors have (significant) biases
  - Real sensors have significant skewness (third moment) and kurtosis (fourth moment)
    - A great deal of information is contained in the tails of the distribution
- Significant sensor testing needs to be performed to fully characterize a sensor and determine its error characteristics
- *Measurement editing* is performed on the innovations process $(\eta_{i_k} = Y_{i_k} - h_i(\hat{\mathbf{X}}_k^-)$ with variance $V_{i_k} = \mathbf{H}_{i_k}\mathbf{P}_k^-\mathbf{H}_{i_k}^T + R_{i_k})$
  - Don't edit out measurements that are greater than $3V_{i_k}$
  - We process measurements that are up to $6V_{i_k}$

Practical Matters – Dealing with Measurement Latency

- Measurements aren't so polite as to be time-tagged or to arrive at the major cycle of the navigation filter (t_k)
- Therefore, we need to process the measurements at the time they are taken, assuming that the measurements are not too latent
    - Provided they are less than (say) 3 seconds latent
- The state is propagated back to the measurement time using, say, a first-order integrator

$$\mathbf{X}_m = \mathbf{X}_k + \mathbf{f}(\mathbf{X}_k)\Delta t + \frac{\partial \mathbf{f}}{\partial \mathbf{X}}(\mathbf{X}_k)\mathbf{f}(\mathbf{X}_k)\Delta t^2$$

- The measurement partial mapping is done in much the same way as it was done in 'batch estimation'
    - Map the measurement sensitivity matrix at the time of the measurement($\mathbf{H}(\mathbf{X}_m)$) to the filter time (t_k) using the state transition matrix, $\mathbf{\Phi}(t_m, t_k)$.

Practical Matters – Measurement Underweighting

- Sometimes, when accurate measurements are introduced to a state which isn't all that accurate, filter instability results
- There are several ways to handle this
  - Second-order Kalman Filters
  - Sigma Point Kalman Filters
  - Measurement Underweighting
- Since Apollo, measurement underweighting has been used extensively
- What underweighting does is it slows down the rate that the measurements decrease the covariance
  - It approximates the second-order correction to the covariance matrix
- Underweighting is typically implemented as

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T \left((1+\alpha) \mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1}$$

- The scalar α is a 'tuning' parameter used to get good filter performance

Practical Matters – Filter Tuning (I)

- Regardless of how you slice it, tuning a navigation filter is an 'art'
- There are (at least) two sets of 'knobs' one can turn to tune a filter
  - Process Noise (also called 'State Noise' or 'Plant Noise'), **Q**, the noise on the state dynamics
  - Measurement Noise, **R**
- Filter tuning is performed in the context of Monte Carlo simulations (1000's of runs)
- FIlter designers *begin* with the expected noise parameters
  - Process Noise – the size of the neglected dynamics (*e.g.* a truncated gravity field)
  - Measurement Noise – the sensor manufacturer's noise specifications

Practical Matters – Filter Tuning (II)

- Sensor parameters (such as bias) are modeled as zero-mean Gauss-Markov parameters, x_p, which have two 'tuning' parameters
  - The Steady State Variance ($P_{p_{ss}}$)
  - The Time Constant (τ)

$$
\begin{aligned}
\frac{d}{dt}x_p &= -\frac{1}{\tau_p}x_p + w_p, \quad \text{where } E[w_p(t)w_p(\tau)] = Q_p\delta(t-\tau) \\
Q_p &= 2\frac{P_{p_{ss}}}{\tau_p}
\end{aligned}
$$

- All of these are 'tuned' in the Monte Carlo environment so that
  - The state error remains mostly within the 3-σ bounds of the filter covariance
  - The filter covariance represents the computed sample covariance

Practical Matters – Filter Tuning (III)

- Sometimes the filter designer inadvertently chooses a process noise such that the covariance of the state gets too small
- When this happens, the filter thinks it is very sure of itself – it is **smug**
- The end result is that the filter starts rejecting measurements
  - Never a good thing
- The solution to this problem is to inject enough process noise to keep the filter 'open'
  - This allows the filter to process measurements appropriately
- There are several spacecraft which have experienced problems because the designers have chosen incorrect (too small) process noise
- Of course, this is nothing more than the classic tension between 'stability' and 'performance'

Practical Matters – Invariance to Measurement Ordering

- Because of its nonlinear foundation, the performance of an EKF can be highly dependent on the order in which measurements are processed
  - For example, if a system processes range and bearing measurements, the performance of the EKF will be different if the range is processed first versus if the bearing were processed first
- To remedy this, on Orion we employ a hybrid linear/EKF formulation
  - The state and covariance updates are accumulated in delta state and covariance variables
  - The state and covariance are updated only after all the measurements are processed

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Lessons Learned

“It may not be right, but it is not necessarily wrong!’

A Former NASA Engineer, 2013

Numerical Checking

- It is vital to ensure that the measurement partials ($\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}$) are calculated correctly
    - This is done off-line by means of numerical differences to approximate the derivative
- It should be axiomatic but If you think you have a 'clever' way of reducing throughput, make sure you check it versus a known result
    - More than one filter has been purported and advertized to be a 'Kalman' Filter and flown just to find out that the fundamental equations are incorrect
    - Thankfully, these have not resulted in failures
    - When the 'correct' equations were implemented, performance improved drastically

Preventing a Smug Filter or Filter Divergence

- Any GNC engineer intuitively grasps the trade-off between stability and performance
- In aerospace navigation we balance filter stability with filter performance
- We keep away from filter divergence at all costs, at the expense of filter performance
- We add process noise to keep the filter ’open’, which has the effect of ’slowing’ down the performance of the system.
- Better a slow filter than a divergent one!!!

Getting to a Right Attitude

- Attitude Determination is sometimes part of the navigation subsystem.
  - At JSC, it has been part of the navigation filter during dynamic phases of flight (ascent and entry)
  - At JPL and GSFC, it is not part of the navigation function because 'Navigation' is their way of saying 'Ground-based Navigation'
- Attitude can be represented in a variety of ways
  - Direction Cosine Matrices
  - Euler Angles
  - Quaternions
  - (Modified) Rodrigues Parameters (MRPs)
  - Gibbs Parameters
- At JSC we choose quaternions for attitude and MRPs for attitude errors
- We must be careful because attitude is not a vector space

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Advanced Topics

- The Kalman-Bucy Filter
- The Schmidt-Kalman Consider Filter
- The Kalman Smoother
- Square Root and Matrix Factorization Techniques
  - Potter Square Root Filter (Apollo)
  - Triangular Square Root Filters
  - UDU Filter (Orion)
- Nonlinear Filters
  - Second-Order Kalman Filters
  - Sigma Point Kalman Filters
  - Particle Filters
  - Entropy Based / Bayesian Inference Filters
- Linear Covariance Analysis

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Conclusions

- Kalman Filtering and Least Squares Estimation are at the heart of the spacecraft navigation
  - Ground-based navigation
  - On-board navigation
- Its purpose is to obtain the 'best' state of the vehicle given a set of measurements and subject to the computational constraints of flight software
- It requires fluency with several disciplines within engineering and mathematics
  - Statistics
  - Numerical Algorithms and Analysis
  - Linear and Nonlinear Analysis
  - Sensor Hardware
- Challenges abound
  - Increased demands on throughput
  - Image-based sensors

To put things in perspective

"I never, never want to be a pioneer . . . Its always best to come in second, when you can look at all the mistakes the pioneers made and then take advantage of them."

Seymour Cray

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