Hyperbolic Trajectory Definition & Sampling

1. Introduction

During the process of interplanetary trajectory design, it is often necessary to fully define a planet*-centered departure or arrival unperturbed hyperbolic reference trajectory from the following input.

$\mu \equiv$ planet reduced mass
$\boldsymbol{N} \equiv$ direction toward the planet north (in the positive right-handed sense) rotational pole, expressed as a planet-centered unit vector with respect to the inertial coordinate system (INR) used consistently throughout this document†
$\boldsymbol{v}_\infty \equiv$ planet-relative INR asymptotic velocity accompanied by its departure or arrival context and whether planet-centered hyperbolic motion is to be prograde or retrograde with respect to $\boldsymbol{N}$
$r_P \equiv$ planet-centered trajectory periapsis distance
$\delta_P \equiv$ planet-centered trajectory declination at periapsis reckoned with respect to the equatorial plane whose normal is $\boldsymbol{N}$

After the hyperbolic trajectory is defined herein, a means of sampling it to obtain planet-centered INR position $\boldsymbol{r}_S$ and velocity $\boldsymbol{v}_S$ at a specified planet-centered distance r_S will be documented for the hyperbola's inbound and outbound legs. Finally, a numeric example of these trajectory definition and sampling techniques will be documented.

2. Hyperbolic Reference Trajectory Definition

The first task in defining the reference trajectory determines the planet-centered unit vector to periapsis $\boldsymbol{P}$ in INR components. The position vector $r_P\,\boldsymbol{P}$ is constrained by hyperbolic trajectory geometry to lie on the *locus of possible injection points* (LPIP) [1, pp. 113-115]. The LPIP is a small circle of radius β on the planet-centered sphere of radius r_P. With respect to the planet's center, the LPIP's center is at $r_P\,\boldsymbol{C}$, where $\boldsymbol{C}$ = unit{-$\boldsymbol{v}_\infty$} in a departure context, and $\boldsymbol{C}$ = unit{+$\boldsymbol{v}_\infty$} in an arrival context. Figure 1 illustrates LPIP geometry in the context of departure from low Earth orbit (LEO) for Mars. Called the asymptote angle, β is determined by first computing the hyperbolic semi-minor axis b as follows [1, Equations 2.79 and 2.63].

$$b = r_P \sqrt{\frac{2\,\mu}{r_P\,v_\infty^{\,2}} + 1} \tag{1}$$

$$\beta = \mathrm{atan}\left\{\frac{b\,v_\infty^{\,2}}{\mu}\right\} \tag{2}$$

* In this discussion, "planet" may be equated with any body having a reasonably fixed rotation axis in inertial space whose gravity field has reasonable spherical symmetry. The Moon would qualify as such a body.
† $\boldsymbol{N}$ is equivalent to the transposed third row of a standard rotation-nutation-precession (RNP) matrix transforming 3-vectors from INR to a planet-fixed coordinate system with the third vector component being the projection onto $\boldsymbol{N}$.

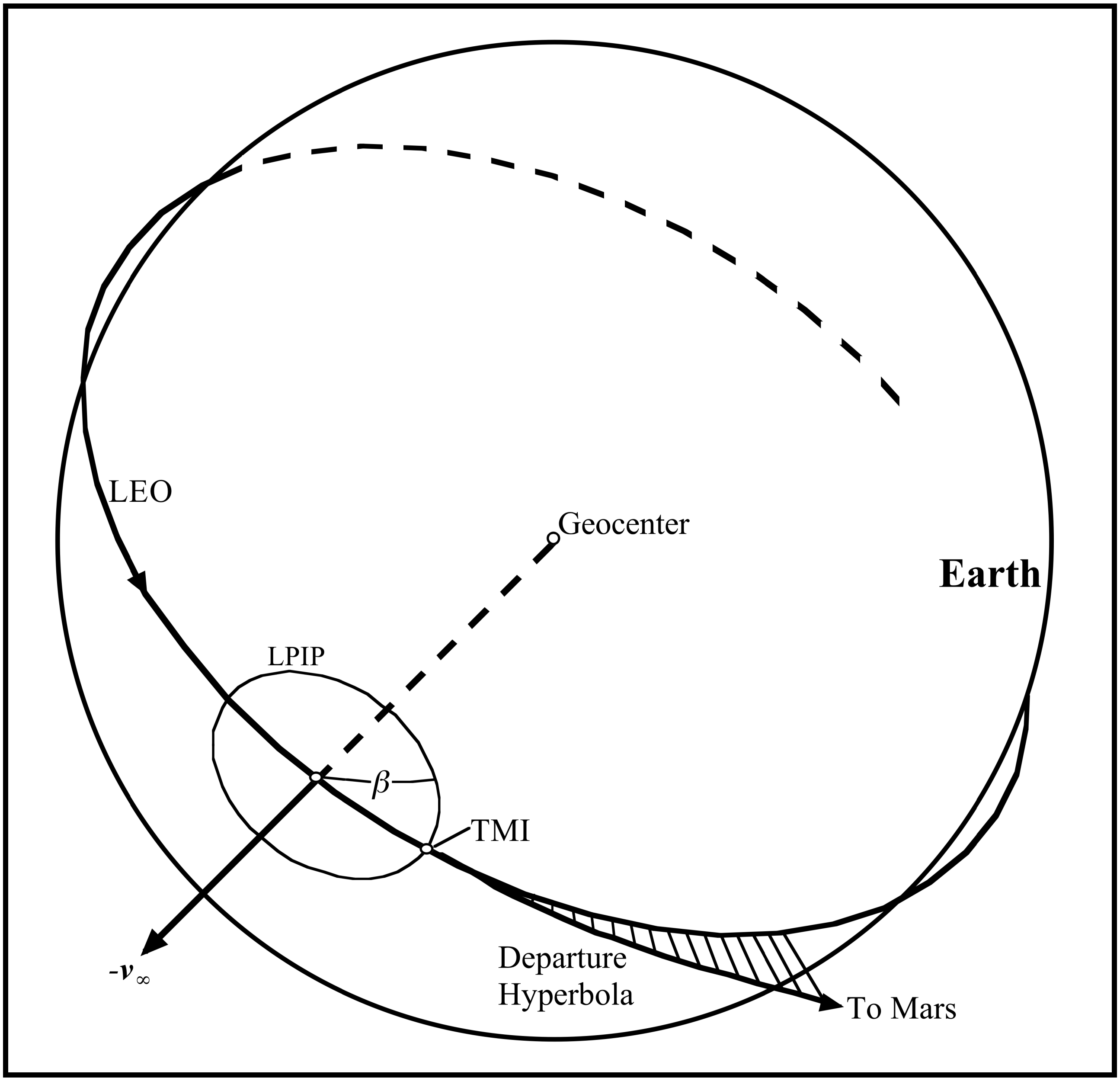

Figure 1. The locus of possible injection points (LPIP) is plotted in the context of departing low Earth orbit (LEO) for Mars. If LEO motion passes through the geocentric $-\boldsymbol{v}_\infty$ direction before coasting through geocentric angle β to reach the LPIP, departure is performed using a minimal, purely prograde trans-Mars injection (TMI) impulse. In an arrival context, a minimal, purely retrograde impulse on the LPIP would establish an orbit intercepting the planet-centered $+\boldsymbol{v}_\infty$ direction after coasting through β.

In obtaining a general expression for LPIP points to satisfy the δ_P constraint, consider an auxiliary planet-centered inertial coordinate system named MEN. This system is defined by unit vector $\boldsymbol{M}$ pointing in the planet's equator to the meridian containing $\boldsymbol{C}$, unit vector $\boldsymbol{E}$ pointing in the planet's equator 90° prograde of $\boldsymbol{M}$, and $\boldsymbol{N}$. Following the right-handed Cartesian convention, $\boldsymbol{M} = \boldsymbol{E} \times \boldsymbol{N}$. The INR-to-MEN transformation is then expressed as follows.

$$M_{INR}^{MEN} = \begin{bmatrix} \boldsymbol{M}^{\mathrm{T}} \equiv \left[\boldsymbol{E} \times \boldsymbol{N}\right]^{\mathrm{T}} \\ \boldsymbol{E}^{\mathrm{T}} \equiv \left[\text{unit}\left\{\boldsymbol{N} \times \boldsymbol{C}\right\}\right]^{\mathrm{T}} \\ \boldsymbol{N}^{\mathrm{T}} \end{bmatrix} \tag{3}$$

The LPIP center's declination is readily computed with Equation 4.

$$\delta_C = \text{asin}\ \{\boldsymbol{N} \bullet \boldsymbol{C}\} \tag{4}$$

A three-angle Euler sequence rotates ***M*** to point from the planet's center toward the direction ***L*** anywhere on the LPIP.

1) Pitch by $-\delta_C$
2) Roll about ***C*** by the unknown angle $-180° \le \phi \le +180°$ required to satisfy the δ_P constraint
3) Yaw by $+\beta$

This Euler sequence is equivalent to the MEN-to-M2L transformation as follows.

$$M_{MEN}^{M2L} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\delta_C & 0 & \sin\delta_C \\ 0 & 1 & 0 \\ -\sin\delta_C & 0 & \cos\delta_C \end{bmatrix}$$

$$M_{MEN}^{M2L} = \begin{bmatrix} \cos\beta & \cos\phi\sin\beta & \sin\phi\sin\beta \\ -\sin\beta & \cos\phi\cos\beta & \sin\phi\cos\beta \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\delta_C & 0 & \sin\delta_C \\ 0 & 1 & 0 \\ -\sin\delta_C & 0 & \cos\delta_C \end{bmatrix}$$

$$M_{MEN}^{M2L} = \begin{bmatrix} \cos\delta_C\cos\beta - \sin\delta_C\sin\phi\sin\beta & \cos\phi\sin\beta & \sin\delta_C\cos\beta + \cos\delta_C\sin\phi\sin\beta \\ -\cos\delta_C\sin\beta - \sin\delta_C\sin\phi\cos\beta & \cos\phi\cos\beta & -\sin\delta_C\sin\beta + \cos\delta_C\sin\phi\cos\beta \\ -\sin\delta_C\cos\phi & -\sin\phi & \cos\delta_C\cos\phi \end{bmatrix}$$

The δ_P constraint is equivalent to finding $\boldsymbol{N} \bullet \boldsymbol{L}$ in the MEN coordinate system. That relationship can be solved for $\sin\phi$.

$$\boldsymbol{N} \bullet \boldsymbol{L} = \sin\delta_P = \boldsymbol{N} \bullet \left[M_{MEN}^{M2L}\right]^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\sin\delta_P = \sin\delta_C\cos\beta + \cos\delta_C\sin\phi\sin\beta$$

$$\sin\phi = \frac{\sin\delta_P - \sin\delta_C\cos\beta}{\cos\delta_C\sin\beta} \tag{5}$$

The right side of Equation 5 must be evaluated to ensure its magnitude does not exceed 1. If it does, no solution for $\boldsymbol{P}$ exists because the specified value of δ_P lies outside the range of declinations spanned by the LPIP. Otherwise, Equation 5 can be solved for ϕ. In doing so, it is essential to be aware the asin function will return values -90° ≤ ϕ_E ≤ +90°. These values are subscripted "E" because they impose a +$\boldsymbol{E}$ component on $\boldsymbol{P}$ (except for degenerate cases where this component is zero at $\phi_E = \pm 90°$). On the LPIP segment "west" of the planet's meridian containing $\boldsymbol{C}$, corresponding ϕ_W angles also exist satisfying Equation 5. In summary, two equations are necessary to obtain both possible ϕ solutions satisfying the δ_P constraint.

$$\phi_E = \operatorname{asin}\left\{\frac{\sin\ \delta_P\ -\ \sin\ \delta_C\ \cos\ \beta}{\cos\ \delta_C\ \sin\ \beta}\right\} \tag{6}$$

$$\phi_W = 180° \operatorname{sign}\{\phi_E\} - \phi_E \tag{7}$$

When assessing whether to select ϕ_E or ϕ_W as the solution leading to $\boldsymbol{P}$, it is necessary to recognize a hyperbolic departure trajectory coasts outward from $\boldsymbol{C}$ to $\boldsymbol{P}$, and a hyperbolic arrival trajectory coasts inward from $\boldsymbol{P}$ to $\boldsymbol{C}$. Mission design context therefore leads to a ϕ_E or ϕ_W selection as summarized in Table 1.

Table 1. Mission design context determines a hyperbolic periapsis location *P* lying east (ϕ_E) or west (ϕ_W) of the planet meridian containing the LPIP's center *C*.

Departure Asymptote	Prograde Motion	ϕ_E
	Retrograde Motion	ϕ_W
Arrival Asymptote	Prograde Motion	ϕ_W
	Retrograde Motion	ϕ_E

With ϕ determined according to Equation 6 or according to Equations 6 and 7 per Table 1 logic, $\boldsymbol{P}$ is computed after transforming $\boldsymbol{L}$ into INR components as follows.

$$\boldsymbol{P} = \left[M_{INR}^{MEN}\right]^{\mathrm{T}} \left[M_{MEN}^{M2L}\right]^{\mathrm{T}} \begin{bmatrix}1\\0\\0\end{bmatrix} = \left[M_{INR}^{MEN}\right]^{\mathrm{T}} \begin{bmatrix}\cos\ \delta_C\ \cos\ \beta\ -\ \sin\ \delta_C\ \sin\ \phi\ \sin\ \beta\\ \cos\ \phi\ \sin\ \beta\\ \sin\ \delta_P\end{bmatrix} \tag{8}$$

The unit vector of specific angular momentum $\boldsymbol{W}$ in the hyperbolic trajectory can now be determined with the aid of mission design context according to Table 2.

Table 2. Mission design context leads to selection of a vector product producing the corresponding specific angular momentum unit vector *W*.

Departure Asymptote	$\boldsymbol{W} = \operatorname{unit}\{\boldsymbol{C} \times \boldsymbol{P}\}$
Arrival Asymptote	$\boldsymbol{W} = \operatorname{unit}\{\boldsymbol{P} \times \boldsymbol{C}\}$

Periapsis speed v_P can be obtained from the energy integral [1, Equation 2.56].

$$v_P = \sqrt{\frac{2\,\mu}{r_P} + v_\infty^{\,2}} \tag{9}$$

And periapsis velocity readily follows.

$$\boldsymbol{v}_P \equiv v_P\,\boldsymbol{Q} = v_P\left(\boldsymbol{W} \times \boldsymbol{P}\right) \tag{10}$$

3. Hyperbolic Reference Trajectory Sampling

With reference trajectory definition parameters computed in the previous section, the trajectory can be sampled at specified planet-centered distance r_S to obtain planet-centered position $\boldsymbol{r}_S$ and velocity $\boldsymbol{v}_S$ in the INR coordinate system. The sampled vectors can be of use as patched conic boundary conditions and sanity checks supporting high fidelity interplanetary trajectory targeting.

Trajectory sampling requires the polar equation for conics be evaluated to obtain true anomaly ν at r_S. This in turn requires the reference trajectory's eccentricity e be computed [1, Equation 2.66], along with its semi-latus rectum p [2, Equation 1.6-1], as follows.

$$e = \frac{1}{\cos\,\beta} \tag{11}$$

$$p = \frac{\left(r_P\,v_P\right)^2}{\mu} \tag{12}$$

In practice, the polar equation for conics is solved for $\cos\,\nu$ [2, Equation 1.5-4].

$$\cos\,\nu = \frac{\frac{p}{r_S} - 1}{e} \tag{13}$$

Sampling context determines the sign (positive for departure; negative for arrival) of $\sin\,\nu$.

$$\sin\,\nu = \pm\sqrt{1 - \cos^2\,\nu} \tag{14}$$

Sampled position and velocity in the planet-centered perifocal (PQW) coordinate system $\boldsymbol{r}_S'$ and $\boldsymbol{v}_S'$ are then computed [2, Equations 2.5-1 and 2.5-4].

$$\boldsymbol{r}_S' = r_S \cos\,\nu\,\boldsymbol{P} + r_S \sin\,\nu\,\boldsymbol{Q} \tag{15}$$

$$\boldsymbol{v}_S' = \sqrt{\frac{\mu}{p}}\left[-\sin\,\nu\,\boldsymbol{P} + \left(e + \cos\,\nu\right)\boldsymbol{Q}\right] \tag{16}$$

Sampled PQW vectors are transformed to INR using calculations from the previous section.

$$M_{INR}^{PQW} = \begin{bmatrix} \boldsymbol{P}^{\mathrm{T}} \\ \boldsymbol{Q}^{\mathrm{T}} \\ \boldsymbol{W}^{\mathrm{T}} \end{bmatrix} \tag{17}$$

$$\boldsymbol{r}_S = \left[M_{INR}^{PQR} \right]^{\mathrm{T}} \boldsymbol{r}_S{}' \tag{18}$$

$$\boldsymbol{v}_S = \left[M_{INR}^{PQR} \right]^{\mathrm{T}} \boldsymbol{v}_S{}' \tag{19}$$

4. Hyperbolic Trajectory Definition & Sampling Numeric Example

The ensuing example is in the context of a prograde arrival at Mars. Trajectory design input values are exact, not truncated or rounded, ensuring consistent output values to the limit of reported precision.

μ = 42,828.3 km^3/s^2

$$\boldsymbol{N} = \begin{bmatrix} +0.446129 \\ -0.406574 \\ +0.797287 \end{bmatrix}$$

$$\boldsymbol{v}_\infty = \begin{bmatrix} -0.567736 \\ +3.569437 \\ +0.565073 \end{bmatrix} \text{ km/s}$$

r_P = 3774 km

δ_P = +2.5°

b = 6196.699 km (1)

β = 62.686° (2)

$$\boldsymbol{C} = \text{unit}\left\{+\boldsymbol{v}_\infty\right\} = \begin{bmatrix} -0.155195 \\ +0.975733 \\ +0.154467 \end{bmatrix}$$

$$M_{INR}^{MEN} = \begin{bmatrix} -0.002413 & +0.890305 & +0.455358 \\ -0.894965 & -0.205072 & +0.396210 \\ +0.446129 & -0.406574 & +0.797287 \end{bmatrix} \tag{3}$$

δ_C = -20.047° (4)

$\sin\phi$ = +0.240713 (5)

ϕ_W = +166.071° (6 & 7)

$$\boldsymbol{P} = \begin{bmatrix} +0.790041 \\ +0.608170 \\ -0.077230 \end{bmatrix} \tag{8}$$

$$\boldsymbol{W} = \begin{bmatrix} +0.190543 \\ -0.123860 \\ +0.973834 \end{bmatrix} \qquad \text{(Table 2)}$$

$$v_P = 6.006581 \text{ km/s} \qquad (9)$$

$$\boldsymbol{Q} = \begin{bmatrix} -0.582691 \\ +0.784084 \\ +0.213737 \end{bmatrix} \qquad (10)$$

$r_S = 7500$ km

$$e = 2.179258 \qquad (11)$$

$$p = 11{,}998.518 \text{ km} \qquad (12)$$

$$\cos \nu = +0.275232 \qquad (13)$$

$$\sin \nu = -0.961378 \qquad (14)$$

$$\boldsymbol{r}_S' = \begin{bmatrix} +2064.243 \\ -7210.333 \\ 0 \end{bmatrix} \text{ km} \qquad (15)$$

$$\boldsymbol{v}_S' = \begin{bmatrix} +1.816334 \\ +4.637275 \\ 0 \end{bmatrix} \text{ km/s} \qquad (16)$$

$$\boldsymbol{r}_S = \begin{bmatrix} +5832.233 \\ -4398.095 \\ -1700.535 \end{bmatrix} \text{ km} \qquad (18)$$

$$\boldsymbol{v}_S = \begin{bmatrix} -1.267121 \\ +4.740654 \\ +0.850881 \end{bmatrix} \text{km/s} \qquad (19)$$

References

[1] Brown, C. D., *Spacecraft Mission Design*, American Institute of Aeronautics and Astronautics, Inc., 1992.

[2] Bate, R. R., Mueller, D. D., and White, J. E., *Fundamentals of Astrodynamics*, Dover Publications, Inc., 1971.