State Vector Reference Body Transformations Throughout The Solar System

Introduction

One of the more mind-boggling programming challenges encountered during interplanetary trajectory design software development is how to transform a specified inertial position and velocity state vector for Body B with respect to Body A, denoted V_A^B, such that it is instead centered on Body C to become V_C^B. Note the transformation, expressed as $V_A^B \rightarrow V_C^B$, only changes the origin of the inertial coordinate system in which both state vectors are expressed. No rotations of this coordinate system are performed. Consequently, this transformation is equivalent to the Equation 1 state vector addition.

$$V_A^B \rightarrow V_C^B = V_A^B + V_C^A \tag{1}$$

Transformations considered by this paper will be called Galilean because only the inertial coordinate system's origin is changed and Special/General Relativity is ignored. In many Galilean transformation definitions, a further condition is constant velocity between the two origins over time. This condition is not generally satisfied between bodies in the Solar System because they undergo differing inertial accelerations. To avoid this violation, any transformation considered herein is conducted with respect to a single specified epoch.

Galilean transformations are of great utility to interplanetary trajectory design. For example, consider a spacecraft (S/C) departing Earth for interplanetary cruise. At some point with the spacecraft about a million km from Earth, a $V_{Earth}^{S/C} \rightarrow V_{Sun}^{S/C}$ transformation would be performed. Perturbing Earth gravity accelerations on the spacecraft's subsequent heliocentric trajectory would then act in the direction of position components given by the transformation $V_{Sun}^{S/C} \rightarrow -V_{Earth}^{S/C}$.

In the context of V_A^B notation, and to underscore the vector nature of a state, this paper's narrative will refer to a subscripted body as the "tail" body and to a superscripted body as the "head" body. When using state vectors from the *Horizons* ephemeris server[1] or the Spacecraft Planet Instrument C-matrix Events (SPICE) information system[2], note that a head body is equivalent to the "target" body and a tail body is equivalent to the "observer" body.

Galilean transformations in this paper comply with a trajectory design ephemeris operations concept developed by the author from experience working with *Horizons* ephemerides and SPICE software since the late 1990s. This concept imports text-based, short-term, typically single-use ephemerides from *Horizons* in which state vectors for spacecraft, asteroids, moons, comets, and other bodies of special interest are stored. Along with these imports, concurrent

[1] This facility may be invoked using links and documentation available at https://ssd.jpl.nasa.gov/?horizons (accessed 16 August 2018).
[2] Software and documentation for SPICE may be downloaded at https://naif.jpl.nasa.gov/naif/ (accessed 16 August 2018).

ephemerides for the Sun, Moon, planets, or Pluto can be created using SPICE software to access a Spacecraft and Planet Kernel (SPK) binary file spanning a relatively protracted time interval.[3]

Method

Under SPICE conventions[4], any body in the Solar System can be assigned an integer ID code. These conventions are summarized in Table 1. Note multiple Table 1 conventions are becoming saturated as the bodies they govern proliferate. Substantial revisions can be expected.

Table 1. Integer ID code assignments for Solar System bodies according to SPICE conventions.

SPICE ID	Body
0	Solar System Barycenter (SSB)
1 through 9	Planetary system barycenters from Mercury through Neptune[5]
10	Sun
199, 299, …, 999	Planetary centers from Mercury through Neptune[6]
n01, n02, …, n98	Moons for systems n = 1 through 9
-100,000 < ID < 0	Interplanetary spacecraft[7]
ID < -100,000	Earth-orbiting spacecraft[8]
1,000,000 < ID < 2,000,000	Comets
2,000,000 < ID < 50,000,000	Asteroids[9]
50,000,000 < ID < 50,000,024	Comet Shoemaker-Levy 9 fragments

When accessed by SPICE utility application `BRIEF`, the SPK file pertinent to this paper's Galilean transformations generates output reproduced in Table 2. In the `Bodies` section of this output, a head body SPICE ID (headID) is followed by "`w.r.t.`" and the corresponding tail body SPICE ID (tailID) for a set of state vector interpolation data. With SPICE software, each dataset generates valid state vectors for the corresponding headID/tailID from epoch 1950.0 to epoch 2100.0 in barycentric dynamical time (TDB).

[3] These text-based *Horizons* imports could be replaced with dedicated SPK binary files optionally concatenated into other such files. But some small bodies are transient in nature. Furthermore, a newly discovered small body is associated with rapidly evolving uncertainty in predicted position as additional observations are accumulated. Consequently, text-based ephemeris imports are the adopted alternative due to their straightforward reconfiguration and traceability.

[4] Reference the SPICE Toolkit download's `doc/naif_ids.req` text file. The toolkit may be downloaded at https://naif.jpl.nasa.gov/naif/toolkit.html (accessed 24 August 2018).

[5] The SPICE ID 9 is assigned to the Pluto system's barycenter.

[6] The SPICE ID 999 is assigned to the center of Pluto.

[7] An interplanetary spacecraft's SPICE ID is typically the negative of its assigned Deep Space Network ID.

[8] An Earth-orbiting spacecraft's SPICE ID is -100,000 minus its assigned NORAD ID. International Space Station has SPICE ID -125,544.

[9] An asteroid's SPICE ID is 2,000,000 plus its JPL catalog number. The asteroid (1) Ceres has SPICE ID 2,000,001.

Table 2. Contents of the SPK file supporting this paper's ephemeris operations concept are summarized by output from the SPICE utility application `BRIEF`.

```
BRIEF -- Version 4.0.0, September 8, 2010 -- Toolkit Version N0066


Summary for: DE431_PLNCNT_1950-2100.bsp

Bodies: MERCURY BARYCENTER (1) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        VENUS BARYCENTER (2) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        EARTH BARYCENTER (3) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        MARS BARYCENTER (4) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        JUPITER BARYCENTER (5) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        SATURN BARYCENTER (6) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        URANUS BARYCENTER (7) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        NEPTUNE BARYCENTER (8) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        PLUTO BARYCENTER (9) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        SUN (10) w.r.t. SOLAR SYSTEM BARYCENTER (0)
        MERCURY (199) w.r.t. MERCURY BARYCENTER (1)
        VENUS (299) w.r.t. VENUS BARYCENTER (2)
        MOON (301) w.r.t. EARTH BARYCENTER (3)
        EARTH (399) w.r.t. EARTH BARYCENTER (3)
        MARS (499) w.r.t. MARS BARYCENTER (4)
        JUPITER (599) w.r.t. JUPITER BARYCENTER (5)
        SATURN (699) w.r.t. SATURN BARYCENTER (6)
        URANUS (799) w.r.t. URANUS BARYCENTER (7)
        NEPTUNE (899) w.r.t. NEPTUNE BARYCENTER (8)
        PLUTO (999) w.r.t. PLUTO BARYCENTER (9)
        Start of Interval (ET)              End of Interval (ET)
        -----------------------------       -----------------------------
        1950 JAN 01 00:00:00.000            2100 JAN 01 00:00:00.000
```

The Table 2 SPK data serve as the "trunk" and main "branches" in a tree-like topology encompassing core Galilean transformations throughout the Solar System. Branches supplementing the SPK file's tree can be added by importing a *Horizons* ephemeris such that the imported tailID is zero (the SSB) or one of the file's headID values. Additional imports can further extend the tree, provided each has a tailID matching a headID somewhere in the tree. Cumulative applications of Equation 1 are used to reconstruct state vector branches in the tree, ultimately computing SSB-centered state vectors with tailID = 0. A portion of the SPK-resident tree described in Table 2 is illustrated in Figure 1 with each **green** arrow representing a state vector. The **orange** arrow denotes a state vector stored in an imported Moon-centered Lunar Reconnaissance Orbiter (LRO) ephemeris "grafted" onto the tree.

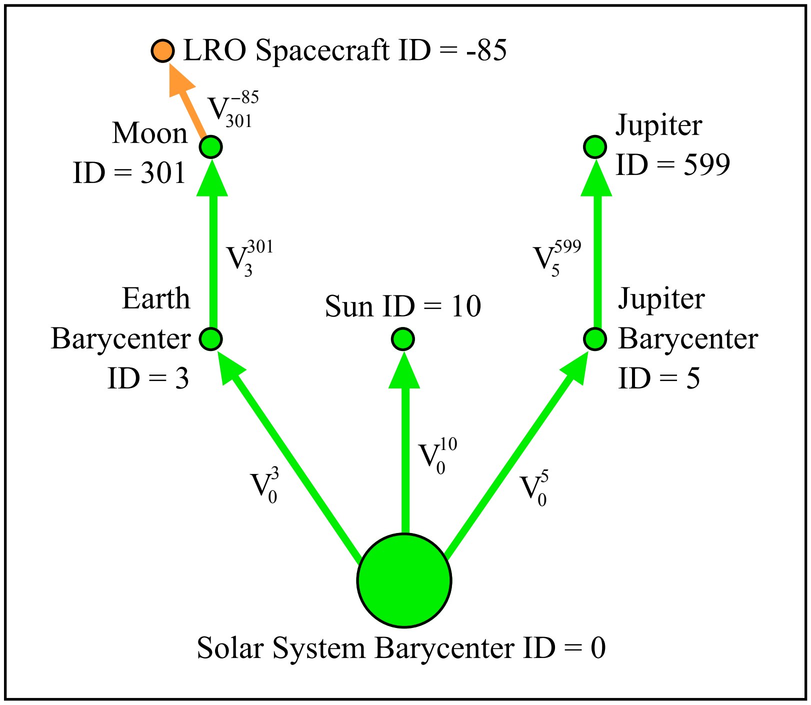

Figure 1. A tree-like topology, relating the Solar System's components through Galilean transformations of state vectors (green arrows), is illustrated for some SPICE IDs in Table 2. The LRO spacecraft has been grafted onto this tree (orange dot) by importing its Moon-centered ephemeris from *Horizons*.[10] Note this diagram is completely relational in nature and not to scale.

When tasked to produce a state vector V_B^A, the standard strategy is to iteratively apply Equation 1 to the initial Body A state vector until its tailID = 0. While thus reconstructing the branch for Body A's headID in the transformation tree, a tailID corresponding to Body B may be encountered serendipitously. If so, the task is complete.

In other less fortuitous cases, Body B in a V_B^A state vector computation lies on a different "secondary" transformation tree branch from the "primary" branch occupied by Body A. In those instances, reconstruction first produces a primary state vector P_0^A for Body A with respect to the SSB. An independent branch reconstruction then produces a secondary SSB-centered state vector S_0^B for Body B whose headID is the desired tailID for Body A's state vector. The specified state vector is then computed with Equation 2.

$$V_B^A = P_0^A - S_0^B \qquad (2)$$

For instance, consider a task to compute solar illumination on the outer martian moon Deimos (SPICE ID = 402). Completion of this task requires an illumination vector equivalent to the position components of V_{10}^{402}. The only imported ephemeris required to produce V_{10}^{402} is one for

[10] This diagram is patterned after a similar example of bodies pertinent to the *Galileo* spacecraft's mission in the SPICE Toolkit download's `doc/spk.req` text file. The toolkit may be downloaded at https://naif.jpl.nasa.gov/naif/toolkit.html (accessed 24 August 2018).

Deimos centered on Mars covering the TDB epoch of interest. Applying Equation 1 to the primary transformation branch occupied by Deimos produces the sequence $P_{499}^{402} \rightarrow P_{4}^{402} \rightarrow P_{0}^{402}$ (per Table 2, tailIDs 499 = Mars and 4 = Mars barycenter). The secondary branch immediately contributes S_{0}^{10} without any transformations. Equation 2 then produces the desired state with $V_{10}^{402} = P_{0}^{402} - S_{0}^{10}$.

Essential to Galilean transformations between ephemerides is an accurate interpolator using TDB epoch as its argument. A 6-point Lagrange interpolator, as documented by Meeus[11], is employed to produce results documented in the following section.

Numeric Example

As illustrated by Figure 1, an object deeply embedded in the Solar System's Galilean transformation tree is the LRO spacecraft with SPICE ID = -85. As of this writing, stable as-flown trajectory data for LRO are available on *Horizons* from shortly after its launch on 18 June 2009 through its lunar orbit insertion 4 days later until 15 March 2018. This example first obtains a Jupiter-centered LRO state vector V_{599}^{-85} on 28 February 2018 at 12:00:00 TDB in the Earth mean equator and equinox of epoch J2000.0 coordinate system (J2K). The result, which requires no Lagrange interpolation, is compared with a similar computation by *Horizons*. A second J2K V_{599}^{-85} state vector is produced 0.0083 d later than the first on 28 February 2018 at 12:11:57.120 TDB such that Lagrange interpolation is required. When compared with synchronized *Horizons* data, this later state vector demonstrates the degree of precision degradation arising from 6-point Lagrange interpolation. Note both V_{599}^{-85} state vectors are *geometric* in that they do not incorporate the effects of planetary aberration or stellar aberration.[12]

This task requires an LRO ephemeris be imported from *Horizons*. Circa late February 2018, *Horizons* indicates LRO's Moon-centered trajectory has a low eccentricity near 0.002 and a period near 117 min. The 6-point Lagrange interpolator is reasonably accurate when operating with 15 state vectors in a single nearly circular orbit's period, suggesting the imported ephemeris have 8 min between consecutive epochs. Starting the imported ephemeris at 11:36 TDB and ending it at 12:48 TDB provides adequate access to P_{301}^{-85} states initiating primary branch reconstructions. To achieve maximum interpolation accuracy, it is essential the 12:11:57.120 TDB epoch lie between epochs 3 and 4 in the 6-state Lagrange interpolation table. Imported ephemeris sequencing is summarized in Table 3.

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[11] Reference J. Meeus, *Astronomical Algorithms*, Willmann-Bell, Inc., 1991, pp. 32-33.

[12] Planetary aberration shifts the geometric state vector of the head body as seen from the tail body according to the finite speed of light. Stellar aberration contributes another shift to this state vector according to the tail body's SSB-centered velocity. These shifts produce the *apparent* state vector of the head body as seen from the tail body. Both aberration shifts can be computed using Galilean transformations as documented herein. Reference the SPICE Toolkit download's `doc/abcorr.req` text file. The toolkit may be downloaded at https://naif.jpl.nasa.gov/naif/toolkit.html (accessed 24 August 2018).

Table 3. A chronology of Moon-centered state vectors appearing in the imported LRO ephemeris follows. These are supplemented with a state vector inferred from 6 of them using a Lagrange interpolator.

28 Feb 2018 TDB Epoch	Ephemeris State Index	Lagrange Table State Index	Comment
11:36:00	1		
11:44:00	2		
11:52:00	3	1	
12:00:00	4	2	No-Lagrange P_{301}^{-85}
12:08:00	5	3	
12:11:57.120			Lagrange-interpolated P_{301}^{-85}
12:16:00	6	4	
12:24:00	7	5	
12:32:00	8	6	
12:40:00	9		
12:48:00	10		

Deviations between this paper's methodology and *Horizons* arising in corresponding J2K V_{599}^{-85} position and velocity vectors are quantified by three parameters. The first of these is ΔRSS, the deviation's vector difference magnitude. Angular deviation between the two vectors is quantified by the $\Delta\theta$ parameter. Finally, Δr is the deviation's signed radial component in the sense "this paper minus *Horizons*". Results appear in Table 4.

Table 4. Deviations between Jupiter-centered LRO state vectors as computed by this paper's methodology and by *Horizons* are summarized with three parameters defined in the foregoing narrative.

28 Feb 2018 TDB Epoch	Vector Components	ΔRSS (m or m/s)	$\Delta\theta$ (deg)	Δr (m or m/s)
12:00:00	Position	0.065	0.000000854	-0.006437302
	Velocity	0.000058	0	-0.000030287
12:11:57.120	Position	802.432	0.000001479	+476.613
	Velocity	0.089765	0.000159596	+0.040021172

Conclusion

An ephemeris operations concept, together with Galilean transformations and associated logic, have been documented with which to relate the position and velocity of any object in the Solar System to any other object therein. Although to some degree limited by ephemeris interpolation accuracy, this capability has been shown to be in close agreement with JPL's *Horizons* ephemeris server.